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ON CONSTRUCTION OF OPTIMAL MIXED-LEVEL SUPERSATURATED DESIGNS

BY FASHENG SUN[1], DENNIS K. J. LIN AND MIN-QIAN LIU[1]

Northeast Normal University, Pennsylvania State University and Nankai University

Supersaturated design (SSD) has received much recent interest because of its potential in factor screening experiments. In this paper, we provide equivalent conditions for two columns to be fully aliased and consequently propose methods for constructing $E(f_{\mathrm{NOD}})$- and χ^2-optimal mixed-level SSDs without fully aliased columns, via equidistant designs and difference matrices. The methods can be easily performed and many new optimal mixed-level SSDs have been obtained. Furthermore, it is proved that the nonorthogonality between columns of the resulting design is well controlled by the source designs. A rather complete list of newly generated optimal mixed-level SSDs are tabulated for practical use.

1. Introduction. The supersaturated design (SSD) is a factorial design in which the number of runs is not sufficient to estimate all the main effects. Such designs are useful when the experiment is expensive, the number of factors is large, and only a few significant factors need to be identified in a relatively small number of experimental runs. Booth and Cox (1962) first examined these designs systematically and proposed the $E(s^2)$ criterion. However, such designs were not further studied until the appearance of the work by Lin (1993, 1995), Wu (1993), Tang and Wu (1997) and Cheng and Tang (2001). Research on mixed-level SSDs includes the early work by Fang, Lin and Liu (2000, 2003) who proposed the $E(f_{\mathrm{NOD}})$ criterion and the FSOA method for constructing mixed-level SSDs, and work by Yamada and Matsui (2002) and Yamada and Lin (2002) who used χ^2 to evaluate mixed-level SSDs. Recent work on mixed-level SSDs includes Xu (2003), Fang et al. (2004a), Li, Liu and Zhang (2004), Xu and Wu (2005), Koukouvinos and Mantas (2005), Liu, Fang and Hickernell (2006), Yamada et al. (2006), Ai, Fang and He (2007), Tang et al. (2007), Chen and Liu (2008a, 2008b), Liu and Lin (2009), Liu and Cai (2009) and Liu and Zhang (2009).

This paper proposes some methods for constructing $E(f_{\mathrm{NOD}})$- and χ^2-optimal mixed-level SSDs without fully aliased columns, and with a control on the

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nonorthogonality. A large number of optimal designs is obtained. The remainder of this paper is organized as follows. Section 2 provides relevant notation and definitions. In Section 3, we propose the general construction methods for mixed-level SSDs along with illustrative examples. Discussions on the nonorthogonality of the resulting designs are given in Section 4. In Section 5, a review of the existing methods for mixed-level SSDs and comparisons with the current methods are made, and some concluding remarks are provided. For coherence of presentation, all proofs are placed in Appendix A and newly constructed designs are tabulated in Appendix B.

2. Preliminaries. A mixed-level design that has n runs and m factors with $q_1, \ldots, q_m$ levels, respectively, is denoted by $F(n, q_1 \cdots q_m)$. When $\sum_{j=1}^{m}(q_j - 1) = n - 1$, the design is called a saturated design, and when $\sum_{j=1}^{m}(q_j - 1) > n - 1$, the design is called a supersaturated design (SSD). An $F(n, q_1 \cdots q_m)$ can be expressed as an $n \times m$ matrix $F = (f_{ij})$. When some q_j's are equal, we use the notation $F(n, q_1^{r_1} \cdots q_l^{r_l})$ indicating r_i factors having q_i levels, $i = 1, \ldots, l$. If all the q_j's are equal, the design is said to be symmetrical and denoted by $F(n, q^m)$. Let f_i be the ith row of an $F(n, q_1 \cdots q_m)$ and f^j be the jth column which takes values from a set of q_j symbols $\{0, \ldots, q_j - 1\}$. If each column f^j is balanced, that is, it contains the q_j symbols equally often, then we say F is a balanced design. Throughout this paper, we only consider balanced designs. Two columns are called fully aliased if one column can be obtained from the other by permuting levels; and called orthogonal if all possible level-combinations for these two columns appear equal number of times. An $F(n, q_1 \cdots q_m)$ is called an orthogonal array of strength two, denoted by $L_n(q^m)$ for the symmetrical case, if all pairs of columns of this design are orthogonal.

The set of residues modulo a prime number p, $\{0, 1, \ldots, p-1\}$, forms a field of p elements under addition and multiplication modulo p, which is called a Galois field and denoted by $\mathrm{GF}(p)$. Note that the order of a Galois field must be a prime power. A Galois field of order $q = p^u$ for any prime p and any positive integer u can be obtained as follows. Let $g(x) = b_0 + b_1 x + \cdots + b_u x^u$ be an irreducible polynomial of degree u, where $b_j \in \mathrm{GF}(p)$ and $b_u = 1$. Then the set of all polynomials of degree $u - 1$ or lower, $\{a_0 + a_1 x + \cdots + a_{u-1} x^{u-1} | a_j \in \mathrm{GF}(p)\}$, is a Galois field $\mathrm{GF}(q)$ of order $q = p^u$ under addition and multiplication of polynomials modulo $g(x)$. For any polynomial $f(x)$ with coefficients from $\mathrm{GF}(p)$, there exist unique polynomials $q(x)$ and $r(x)$ such that $f(x) = q(x)g(x) + r(x)$, where the degree of $r(x)$ is lower than u. This $r(x)$ is the residue of $f(x)$ modulo $g(x)$, which is usually written as $f(x) = r(x) (\mathrm{mod}\, g(x))$.

A difference matrix, denoted by $D(rq, c, q)$, is an $rq \times c$ array with entries from a finite Abelian group $(\mathcal{A}, +)$ with q elements such that each element of $\mathcal{A}$ appears equally often in the vector of difference between any two columns of the array [Bose and Bush (1952)]. Note that if A is an $L_{rq}(q^c)$, then it is also a difference matrix. A difference matrix $D(rq, c, q)$ with $c > 1$ is said to be normalized, denoted by $\mathrm{ND}(rq, c, q)$, if its first column consists of all zeros. In fact, for any difference matrix D, if we subtract the first column from any column, then we can obtain a normalized difference matrix.

For a scalar a and a matrix A, let $a + A$ denote the element-wise sum of a and A. For any two matrices $A = (a_{ij})$ of order $r \times s$ and B of order $u \times v$, their Kronecker sum and Kronecker product are defined to be

$$A \oplus B = \begin{pmatrix} a_{11}+B & \cdots & a_{1s}+B \\ \cdots & \cdots & \cdots \\ a_{r1}+B & \cdots & a_{rs}+B \end{pmatrix} \quad \text{and} \quad A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1s}B \\ \cdots & \cdots & \cdots \\ a_{r1}B & \cdots & a_{rs}B \end{pmatrix},$$

respectively. Here, we use "$+_{\mathcal{A}}$" and "$\oplus_{\mathcal{A}}$" to denote the sum and Kronecker sum defined on $\mathcal{A}$, respectively.

For a design $F = (f_{ij})_{n\times m}$, let

$$\lambda_{ij}(F) = \sum_{k=1}^{m} \delta_{ij}^{(k)} \quad \text{and} \quad \omega_{ij}(F) = \sum_{k=1}^{m} q_k \delta_{ij}^{(k)},$$

where $\delta_{ij}^{(k)} = 1$ if $f_{ik} = f_{jk}$, and 0 otherwise. Then $\lambda_{ij}(F)$ and $\omega_{ij}(F)$ are called the coincidence number and natural weighted coincidence number between rows f_i and f_j, respectively. A design with equal coincidence numbers between different rows is called an equidistant design. From Mukerjee and Wu (1995), a saturated $L_n(q^m)$ is an equidistant design with

$$\lambda_{ij}(F) = \frac{m-1}{q} \quad \text{and} \quad \omega_{ij}(F) = m-1 \qquad \text{for } i \neq j. \tag{1}$$

The $E(f_{\text{NOD}})$ criterion proposed by Fang, Lin and Liu (2000, 2003) is defined to minimize

$$E(f_{\text{NOD}}) = \frac{2}{m(m-1)} \sum_{1\le i<j\le m} f_{\text{NOD}}(f^i, f^j),$$

where

$$f_{\text{NOD}}(f^i, f^j) = \sum_{a=0}^{q_i-1} \sum_{b=0}^{q_j-1} \left(n_{ab}(f^i, f^j) - \frac{n}{q_i q_j} \right)^2,$$

$n_{ab}(f^i, f^j)$ is the number of (a, b)-pairs in (f^i, f^j), and $n/(q_i q_j)$ stands for the average frequency of level-combinations in (f^i, f^j). Here, the subscript "NOD" stands for *nonorthogonality of the design*. The $f_{\text{NOD}}(f^i, f^j)$ value gives a nonorthogonality measure for (f^i, f^j), and columns f^i and f^j are orthogonal if and only if $f_{\text{NOD}}(f^i, f^j) = 0$. It is obvious that F is an orthogonal array if and only if $E(f_{\text{NOD}}) = 0$, that is, $f_{\text{NOD}}(f^i, f^j) = 0$ for all $i, j = 1, \dots, m, i \neq j$. Thus $E(f_{\text{NOD}})$ measures the average nonorthogonality among the columns of F.

Another criterion that is to be minimized was defined by Yamada and Lin (1999) and Yamada and Matsui (2002) as $\chi^2(F) = \sum_{1\le i<j\le m} q_i q_j f_{\text{NOD}}(f^i, f^j)/n$. Obviously, $E(f_{\text{NOD}})$ and $\chi^2(F)$ are equivalent in the symmetrical case. Here, we

adopt both $E(f_{\mathrm{NOD}})$ and $\chi^2(F)$ to evaluate the newly constructed SSDs. There are also some other criteria for assessing mixed-level SSDs [see, e.g., Liu and Lin (2009) for a general review].

The following results, regarding the $E(f_{\mathrm{NOD}})$ and $\chi^2(F)$ optimality criteria of a design, will be needed for our construction methods.

LEMMA 1. (a) [Fang et al. (2004a)]. *If the difference among all coincidence numbers between different rows of design F does not exceed one, then F is $E(f_{\mathrm{NOD}})$-optimal.*

(b) [Li, Liu and Zhang (2004); Liu, Fang and Hickernell (2006)]. *If the natural weighted coincidence numbers between different rows of design F take at most two nearest values, then F is χ^2-optimal.*

3. Proposed construction methods. In this section, we first provide some equivalent conditions for two columns to be fully aliased, then propose methods for constructing $E(f_{\mathrm{NOD}})$- and χ^2-optimal SSDs, and finally study the properties of the resulting designs.

3.1. *Equivalent conditions for two columns to be fully aliased.* An $E(f_{\mathrm{NOD}})$- or χ^2-optimal SSD may contain fully aliased columns, which is undesirable. Let matrix $X_j = (x^j_{st})$ of order $n \times q_j$ be the induced matrix [Fang et al. (2004a)] of the jth column of an $F(n, q_1 \cdots q_m)$, that is, $x^j_{st} = 1$ if the sth element in the jth column is $t-1$, otherwise 0, for $s = 1, \dots, n$, $t = 1, \dots, q_j$ and $j = 1, \dots, m$. The following theorem presents theoretical results concerning the column aliasing that will be used in the construction methods.

THEOREM 1. *Suppose $X_j = (x^j_{st})$ is the induced matrix of a balanced column $f^j = (f_{1j}, \dots, f_{n_j j})'$ with q_j levels, $j = 1, \dots, 4$, and $n_1 = n_3, n_2 = n_4$.*

(a) *For $q_1 = q_2 = q_3 = q_4 = q$ and $\mathcal{A} = \{0, \dots, q-1\}$:*

(i) *f^1 and f^3 are fully aliased if and only if $X_1 X_1' = X_3 X_3'$;*

(ii) *the induced matrix of $f^1 \oplus_{\mathcal{A}} f^2$ is $[(X_2 P_{f_{11}})', \dots, (X_2 P_{f_{n_1 1}})']' = (X_1 \otimes X_2)P$, where $P = (P_0', \dots, P_{q-1}')'$ and P_i is a permutation matrix defined by*

$$i +_{\mathcal{A}} (0, \dots, q-1) = (0, \dots, q-1)P_i', \qquad i = 0, \dots, q-1;$$

(iii) *if $f^1 \oplus_{\mathcal{A}} f^2$ and $f^3 \oplus_{\mathcal{A}} f^4$ are fully aliased, then f^1 is fully aliased with f^3 and f^2 is fully aliased with f^4.*

(b) (i) *The induced matrix of the $q_1 q_2$-level column $q_2(f^1 - \frac{q_1-1}{2}) \oplus (f^2 - \frac{q_2-1}{2}) + \frac{q_1 q_2 - 1}{2}$ is $X_1 \otimes X_2$;*

(ii) *columns* $q_2(f^1 - \frac{q_1-1}{2}) \oplus (f^2 - \frac{q_2-1}{2}) + \frac{q_1q_2-1}{2}$ *and* $q_4(f^3 - \frac{q_3-1}{2}) \oplus (f^4 - \frac{q_4-1}{2}) + \frac{q_3q_4-1}{2}$ *are fully aliased if and only if* f^1 *is fully aliased with* f^3 *and* f^2 *is fully aliased with* f^4;

(iii) *for* $q_3 = q_4 = q$, $q_2(f^1 - \frac{q_1-1}{2}) \oplus (f^2 - \frac{q_2-1}{2}) + \frac{q_1q_2-1}{2}$ *and* $f^3 \oplus_{\mathcal{A}} f^4$ *are not fully aliased in any case.*

3.2. *Construction of optimal symmetrical SSDs.* We next present the methods for constructing $E(f_{\mathrm{NOD}})$- and χ^2-optimal SSDs without fully aliased columns.

THEOREM 2. *Let* D *be an* $\mathrm{ND}(rq, c, q)$ *defined on an Abelian group* $\mathcal{A} = \{0, \dots, q-1\}$ *without identical rows,* F *be an* $F(n, q^m)$ *without fully aliased columns and with constant coincidence numbers, say* λ, *between its different rows, then*:

(a) $F \oplus_{\mathcal{A}} D'$ *is an* $F(cn, q^{rqm})$ *with two different values of coincidence numbers,* mr *and* λrq;

(b) $F \oplus_{\mathcal{A}} D'$ *has no fully aliased columns.*

From Lemma 1, if $|mr - \lambda rq| \leq 1$, then $F \oplus_{\mathcal{A}} D'$ is both $E(f_{\mathrm{NOD}})$- and χ^2-optimal. The following corollary can be directly obtained from Lemma 1, Theorem 2, and equation (1).

COROLLARY 1. *Let* F *be a saturated* $L_n(q^m)$ *and* D *be an* $\mathrm{ND}(q, c, q)$ *without identical rows. Then* $F \oplus_{\mathcal{A}} D'$ *is an* $F(cn, q^{mq})$ *without fully aliased columns and with two different values of coincidence numbers,* m *and* $m-1$, *and thus is both* $E(f_{\mathrm{NOD}})$- *and* χ^2-*optimal.*

From Hedayat, Slone and Stufken (1999), there exist an $L_n(q^m)$ with $n = q^t$ and $m = (n-1)/(q-1)$ and an $\mathrm{ND}(q, q, q)$ without identical rows for any prime power q, thus optimal $F(cq^t, q^{(q^{t+1}-q)/(q-1)})$ designs with coincidence numbers $(q^t-1)/(q-1) - 1$ or $(q^t-1)/(q-1)$ can be constructed from Corollary 1, where c is a positive integer and $c < q$.

EXAMPLE 1. Let F be an $L_9(3^4)$ and D be an $\mathrm{ND}(3, 2, 3)$ (cf. Table 1), then $F \oplus_{\mathcal{A}} D'$ is an $F(18, 3^{12})$ with coincidence numbers 4 and 3 as listed in Table 2, where $\mathcal{A} = \mathrm{GF}(3)$. This new design is an $E(f_{\mathrm{NOD}})$- and χ^2-optimal SSD without fully aliased columns.

3.3. *Construction of optimal SSDs with two different level sizes.* Based on Lemma 1 and Theorem 2, the following theorem can be obtained.

THEOREM 3. *Let* F_i *be an* $F(n_i, q_i^{m_i})$ *with constant coincidence numbers* λ_i, *and no full aliased columns,* $i = 1, 2$. *Let* D *be an* $\mathrm{ND}(rq_1, n_2, q_1)$ *defined on Abelian group* $\mathcal{A}_1 = \{0, \dots, q_1 - 1\}$ *without identical rows. Then* $F =$

TABLE 1
F and D in Example 1

F'									D	
0	0	0	1	1	1	2	2	2	0	0
0	1	2	0	1	2	0	1	2	0	1
0	1	2	1	2	0	2	0	1	0	2
0	2	1	1	0	2	2	1	0		

$(F_1 \oplus_{\mathcal{A}_1} D', 0_{n_1} \oplus F_2)$ *is an* $F(n_1 n_2, q_1^{rm_1 q_1} q_2^{m_2})$ *without full aliased columns. Furthermore*:

(a) *if* $|(\lambda_2 + rm_1) - (m_2 + \lambda_1 r q_1)| \le 1$, *then* F *is* $E(f_{\mathrm{NOD}})$*-optimal*;
(b) *if* $q_2\lambda_2 + q_1 r m_1 = q_2 m_2 + \lambda_1 r q_1^2$, *then* F *is* χ^2*-optimal.*

Next, let us consider two illustrative examples for Theorem 3.

EXAMPLE 2. Let F_1 be an $L_4(2^3)$, F_2 be the $E(f_{\mathrm{NOD}})$-optimal $F(6, 3^5)$ obtained by Fang, Ge and Liu (2004) and D be an ND(8, 6, 2) without identical rows obtained from an $L_8(2^7)$ based on $\mathcal{A} = \mathrm{GF}(2)$. Then $\lambda_1 = \lambda_2 = 1, q_1 = 2, q_2 = 3, r = 4, m_1 = 3$ and $m_2 = 5$ which satisfy the condition that $\lambda_2 + rm_1 = m_2 + \lambda_1 r q_1 = 13$, thus $(F_1 \oplus_{\mathcal{A}} D', 0_4 \oplus F_2)$ is an $E(f_{\mathrm{NOD}})$-optimal $F(24, 2^{24}3^5)$

TABLE 2
The $F(18, 3^{12})$ *constructed in Example* 1

$F \oplus_{\mathcal{A}} D'$											
0	0	0	0	0	0	0	0	0	0	0	0
0	1	2	0	1	2	0	1	2	0	1	2
0	0	0	1	1	1	1	1	1	2	2	2
0	1	2	1	2	0	1	2	0	2	0	1
0	0	0	2	2	2	2	2	2	1	1	1
0	1	2	2	0	1	2	0	1	1	2	0
1	1	1	0	0	0	1	1	1	1	1	1
1	2	0	0	1	2	1	2	0	1	2	0
1	1	1	1	1	1	2	2	2	0	0	0
1	2	0	1	2	0	2	0	1	0	1	2
1	1	1	2	2	2	0	0	0	2	2	2
1	2	0	2	0	1	0	1	2	2	0	1
2	2	2	0	0	0	2	2	2	2	2	2
2	0	1	0	1	2	2	0	1	2	0	1
2	2	2	1	1	1	0	0	0	1	1	1
2	0	1	1	2	0	0	1	2	1	2	0
2	2	2	2	2	2	1	1	1	0	0	0
2	0	1	2	0	1	1	2	0	0	1	2

TABLE 3
F_1, F_2 and D in Example 2

F_1			F_2					D'							
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	1	0	1	1	1	1	0	0	1	1	1	1	0	0
1	0	1	1	0	2	2	1	0	1	0	1	1	0	1	0
1	1	0	1	2	0	1	2	0	1	0	1	0	1	0	1
			2	1	2	0	2	0	0	1	1	0	0	1	1
			2	2	1	2	0	0	1	1	0	0	1	1	0

with constant coincidence numbers 13. The source designs and resulting design are listed in Tables 3 and 4, respectively.

EXAMPLE 3. Let F_1 be an $L_4(2^3)$, F_2 be the $F(6, 3^{10})$ obtained by Georgiou and Koukouvinos (2006) and D be an ND(24, 6, 2) without identical rows ob-

TABLE 4
The $F(24, 2^{24}3^5)$ constructed in Example 2

$F_1 \oplus_{\mathcal{A}} D'$																								$0_4 \oplus F_2$				
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	1	1	1	1
0	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0	0	1	0	1	1	0	1	0	1	0	2	2	1
0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	2	0	1	2
0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	2	1	2	0	2
0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	2	2	1	2	0
0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0
0	0	1	1	1	1	0	0	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	0	1	1	1	1
0	1	0	1	1	0	1	0	1	0	1	0	0	1	0	1	1	0	1	0	0	1	0	1	1	0	2	2	1
0	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	2	0	1	2
0	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	2	1	2	0	2
0	1	1	0	0	1	1	0	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	2	2	1	2	0
1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0
1	1	0	0	0	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0	0	0	1	1	0	1	1	1	1
1	0	1	0	0	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0	0	1	0	1	1	0	2	2	1
1	0	1	0	1	0	1	0	0	1	0	1	0	1	0	1	1	0	1	0	1	0	1	0	1	2	0	1	2
1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1	1	1	0	0	1	1	0	0	2	1	2	0	2
1	0	0	1	1	0	0	1	0	1	1	0	0	1	1	0	1	0	0	1	1	0	0	1	2	2	1	2	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	0	0	1	1	1	1	0	0	0	1	1	1	1
1	0	1	0	0	1	0	1	1	0	1	0	0	1	0	1	0	1	0	1	1	0	1	0	1	0	2	2	1
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0	1	0	1	0	1	0	1	1	2	0	1	2
1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	0	0	1	1	0	0	1	1	2	1	2	0	2
1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	0	1	1	0	0	1	1	0	2	2	1	2	0

tained from an $L_{24}(2^{23})$ based on $\mathcal{A} = \mathrm{GF}(2)$. Then $\lambda_1 = 1, \lambda_2 = 2, q_1 = 2, q_2 = 3, r = 12, m_1 = 3$ and $m_2 = 10$ which satisfy the condition that $q_2\lambda_2 + q_1 r m_1 = q_2 m_2 + \lambda_1 r q_1^2 = 78$, thus $(F_1 \oplus_{\mathcal{A}} D', 0_4 \oplus F_2)$ is a $\chi^2(D)$-optimal $F(24, 2^{72}3^{10})$ with constant natural weighted coincidence numbers 78. Exact details are omitted here but available upon request.

3.4. *Construction of optimal SSDs with three different level sizes.* The next lemma is useful in the upcoming proposed construction method.

LEMMA 2. *Let* $V = \{-\frac{q-1}{2}, -\frac{q-3}{2}, \dots, \frac{q-3}{2}, \frac{q-1}{2}\} = \{0, \dots, q-1\} - \frac{q-1}{2}$ *and* $V_i = (i - \frac{p-1}{2})q + V, i = 0, \dots, p-1$, *then* $V_i \cap V_j = \Phi$ *for* $i \neq j$ *and* $\bigcup_{i=0}^{p-1} V_i = \{-\frac{pq-1}{2}, -\frac{pq-3}{2}, \dots, \frac{pq-3}{2}, \frac{pq-1}{2}\} = \{0, \dots, pq-1\} - \frac{pq-1}{2}$, *where* Φ *is an empty set.*

From this lemma, we can obtain the following theorem in a straightforward manner.

THEOREM 4. *Let* F_i *be an* $F(n_i, q_i^{m_i})$ *with constant coincidence numbers* λ_i, $i = 1, 2$, *then* $q_2(F_1 - \frac{q_1-1}{2}) \oplus (F_2 - \frac{q_2-1}{2}) + \frac{q_1q_2-1}{2}$ *is an* $F(n_1n_2, (q_1q_2)^{m_1m_2})$ *with three different values of coincidence numbers* $\lambda_1 m_2, \lambda_2 m_1$ *and* $\lambda_1\lambda_2$.

This theorem, along with Lemma 1 and Theorem 2, leads to the following theorem, which provides another construction method of $E(f_{\mathrm{NOD}})$- and χ^2-optimal SSDs.

THEOREM 5. *Suppose* F_i *is an* $F(n_i, q_i^{m_i})$ *with constant coincidence numbers* λ_i *and no fully aliased columns*, $i = 1, \dots, 4$, D_3 *is an* $\mathrm{ND}(r_3q_3, n_2, q_3)$ *defined on Abelian group* $\mathcal{A}_3 = \{0, \dots, q_3 - 1\}$ *without identical rows*, D_4 *is an* $\mathrm{ND}(r_4q_4, n_1, q_4)$ *defined on* $\mathcal{A}_4 = \{0, \dots, q_4 - 1\}$ *without identical rows, and they satisfy* (i) $n_1 = n_3, n_2 = n_4$; (ii) *the first rows of* F_3 *and* F_4 *consist of all zeros*; (iii) *there are no fully aliased columns between* F_3 *and* D_4' *or between* F_4 *and* D_3'. *Then*

$$F = \left[q_2\left(F_1 - \frac{q_1-1}{2}\right) \oplus \left(F_2 - \frac{q_2-1}{2}\right) + \frac{q_1q_2-1}{2},\right.$$
$$\left. F_3 \oplus_{\mathcal{A}_3} D_3', D_4' \oplus_{\mathcal{A}_4} F_4\right] \tag{2}$$

is an $F(n_1n_2, (q_1q_2)^{m_1m_2} q_3^{m_3r_3q_3} q_4^{m_4r_4q_4})$ *without fully aliased columns and*:

(a) *if the difference among three values* $\lambda_2 m_1 + r_3 m_3 + \lambda_4 r_4 q_4$, $\lambda_1\lambda_2 + r_3 m_3 + r_4 m_4$ *and* $\lambda_1 m_2 + \lambda_3 r_3 q_3 + r_4 m_4$ *does not exceed one, then* F *is* $E(f_{\mathrm{NOD}})$*-optimal*;

(b) *if* $q_1q_2\lambda_2 m_1 + q_3 r_3 m_3 + \lambda_4 r_4 q_4^2 = q_1q_2\lambda_1\lambda_2 + q_3 r_3 m_3 + q_4 r_4 m_4 = q_1q_2\lambda_1 m_2 + \lambda_3 r_3 q_3^2 + q_4 r_4 m_4$, *then* F *is* χ^2*-optimal.*

The following two examples serve as illustrations of the construction method in Theorem 5.

EXAMPLE 4. Let F_1 and F_3 be two $L_4(2^3)$'s; F_2 be the $F(6, 2^{10})$ obtained by Liu and Zhang (2000); F_4 be the $F(6, 3^5)$ obtained by Fang, Ge and Liu (2004); D_3 be an ND(12, 6, 2) without identical rows obtained from an $L_{12}(2^{11})$; D_4 be an ND(12, 4, 3) without identical rows; $\mathcal{A}_3 = \mathrm{GF}(2)$ and $\mathcal{A}_4 = \mathrm{GF}(3)$. Suppose the first rows of F_3 and F_4 consist of all zeros. Then based on Theorem 5, $\lambda_1 = \lambda_3 = \lambda_4 = 1, \lambda_2 = 4, m_1 = m_3 = 3, m_2 = 10, m_4 = 5, q_1 = q_2 = q_3 = 2, q_4 = 3, r_3 = 6, r_4 = 4$ and $\lambda_2 m_1 + r_3 m_3 + \lambda_4 r_4 q_4 = \lambda_1\lambda_2 + r_3 m_3 + r_4 m_4 = \lambda_1 m_2 + \lambda_3 r_3 q_3 + r_4 m_4 = 42$. Thus, from (2), we obtain an $E(f_{\mathrm{NOD}})$-optimal $F(24, 4^{30}2^{36}3^{60})$ with constant coincidence numbers 42 and no fully aliased columns.

EXAMPLE 5. Let F_1 and F_3 be two $L_4(2^3)$'s; both F_2 and F_4 be the $F(6, 3^5)$ obtained by Fang, Ge and Liu (2004); D_3 be an ND(24, 6, 2) without identical rows obtained from an $L_{24}(2^{23})$ based on $\mathcal{A}_3 = \mathrm{GF}(2)$ and D_4 be an ND(6, 4, 3) without identical rows based on $\mathcal{A}_4 = \mathrm{GF}(3)$. Suppose the first rows of F_3 and F_4 consist of all zeros. Then $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1, q_1 = q_3 = 2, q_2 = q_4 = 3, r_3 = 12, r_4 = 2, m_1 = m_3 = 3, m_2 = m_4 = 5$, which satisfy the condition that $q_1 q_2 \lambda_2 m_1 + q_3 r_3 m_3 + \lambda_4 r_4 q_4^2 = q_1 q_2 \lambda_1 \lambda_2 + q_3 r_3 m_3 + q_4 r_4 m_4 = q_1 q_2 \lambda_1 m_2 + \lambda_3 r_3 q_3^2 + q_4 r_4 m_4 = 108$. Thus, the design constructed through (2) is a χ^2-optimal $F(24, 6^{15}2^{72}3^{30})$ with constant natural weighted coincidence numbers 108 and no fully aliased columns.

4. Nonorthogonality of the resulting designs. In the previous section, construction methods for $E(f_{\mathrm{NOD}})$- as well as χ^2-optimal SSDs without fully aliased columns are provided. Full aliasing can be viewed as the extreme case of nonorthogonality. In this section, we will investigate nonorthogonality, measured by f_{NOD}, of the resulting designs, and show how it is controlled by the source designs.

THEOREM 6. *Suppose $f^i = (f_{1i}, \dots, f_{n_i i})'$ is a q_i-level balanced column with induced matrix X_i, $\mathcal{A}_i = \{0, \dots, q_i - 1\}$, $i = 1, \dots, 4$, $n_1 = n_3, n_2 = n_4$. Let $h_1 = q_2(f^1 - \frac{q_1-1}{2}) \oplus (f^2 - \frac{q_2-1}{2}) + \frac{q_1q_2-1}{2}$ and $h_2 = q_4(f^3 - \frac{q_3-1}{2}) \oplus (f^4 - \frac{q_4-1}{2}) + \frac{q_3q_4-1}{2}$. Then*:

(a) $f_{\mathrm{NOD}}(h_1, h_2) = f_{\mathrm{NOD}}(f^1, f^3) f_{\mathrm{NOD}}(f^2, f^4) + \frac{n_2^2}{q_2q_4} f_{\mathrm{NOD}}(f^1, f^3) + \frac{n_1^2}{q_1q_3} f_{\mathrm{NOD}}(f^2, f^4)$;

(b) *if $q_1 = q_2, q_3 = q_4$, then*

$$f_{\mathrm{NOD}}(f^1 \oplus_{\mathcal{A}_1} f^2, f^3 \oplus_{\mathcal{A}_3} f^4) \le q_1 q_3 f_{\mathrm{NOD}}(f^1, f^3) f_{\mathrm{NOD}}(f^2, f^4) + \min\{n_2^2 f_{\mathrm{NOD}}(f^1, f^3), n_1^2 f_{\mathrm{NOD}}(f^2, f^4)\},$$

where the equality holds if and only if f^1 *is orthogonal to* f^3 *or* f^2 *is orthogonal to* f^4;

(c) *if* $q_1 = q_2$, *then*

$$f_{\mathrm{NOD}}(f^1 \oplus_{\mathcal{A}_1} f^2, h_2) \leq q_1 f_{\mathrm{NOD}}(f^1, f^3) f_{\mathrm{NOD}}(f^2, f^4) + \min\{n_2^2/q_4 f_{\mathrm{NOD}}(f^1, f^3), n_1^2/q_3 f_{\mathrm{NOD}}(f^2, f^4)\},$$

where the equality holds if and only if f^1 *is orthogonal to* f^3 *or* f^2 *is orthogonal to* f^4.

Theorem 6 shows that the nonorthogonality measured by f_{NOD} of the resulting designs is well controlled by the source designs. If the source designs have small values of f_{NOD}, then the resulting design will also have small values of f_{NOD}. In particular, we have the following.

COROLLARY 2. *Suppose* $f^i = (f_{1i}, \dots, f_{n_i i})'$ *is a* q_i*-level balanced column with induced matrix* X_i, $\mathcal{A}_i = \{0, \dots, q_i - 1\}$, $i = 1, \dots, 4$, $n_1 = n_3, n_2 = n_4$. *Then*:

(a) *if* f^1 *is orthogonal to* f^3 *and* f^2 *is orthogonal to* f^4, *then* $q_2(f^1 - \frac{q_1-1}{2}) \oplus (f^2 - \frac{q_2-1}{2}) + \frac{q_1 q_2 - 1}{2}$ *is orthogonal to* $q_4(f^3 - \frac{q_3-1}{2}) \oplus (f^4 - \frac{q_4-1}{2}) + \frac{q_3 q_4 - 1}{2}$;

(b) *if* $q_1 = q_2$, $q_3 = q_4$ *and* f^1 *is orthogonal to* f^3 *or* f^2 *is orthogonal to* f^4, *then* $f^1 \oplus_{\mathcal{A}_1} f^2$ *is orthogonal to* $f^3 \oplus_{\mathcal{A}_3} f^4$;

(c) *if* $q_1 = q_2$ *and* f^1 *is orthogonal to* f^3 *or* f^2 *is orthogonal to* f^4, *then* $f^1 \oplus_{\mathcal{A}_1} f^2$ *is orthogonal to* $q_4(f^3 - \frac{q_3-1}{2}) \oplus (f^4 - \frac{q_4-1}{2}) + \frac{q_3 q_4 - 1}{2}$.

This corollary indicates that the orthogonality between columns of the source design is maintained in the generated designs.

5. Discussion and concluding remarks. In this paper, we have presented some construction methods for $E(f_{\mathrm{NOD}})$- and χ^2-optimal SSDs. A review of the existing methods for mixed-level SSDs and comparisons with the current methods are summarized below.

(a) Yamada and Matsui (2002) and Yamada and Lin (2002) proposed two methods for constructing mixed-level SSDs consisting of only two- and three-level columns through computer searches. However, their resulting designs have no theoretical support and typically are unable to achieve the lower bound of χ^2-value.

(b) Fang, Lin and Liu (2000, 2003) proposed an FSOA method for constructing $E(f_{\mathrm{NOD}})$-optimal mixed-level SSDs from saturated orthogonal arrays. Li, Liu and Zhang (2004) and Ai, Fang and He (2007) extended the FSOA method to construct χ^2-optimal SSDs. Koukouvinos and Mantas (2005) constructed some $E(f_{\mathrm{NOD}})$-optimal mixed-level SSDs by juxtaposing either a saturated two-level orthogonal array and an $E(f_{\mathrm{NOD}})$-optimal mixed-level SSD, or two $E(f_{\mathrm{NOD}})$-optimal SSDs.

Fang et al. (2004a) and Tang et al. (2007) presented some methods for constructing $E(f_{\mathrm{NOD}})$- and χ^2-optimal mixed-level SSDs, respectively, from given combinatorial designs. There are many constraints on the parameters of saturated orthogonal arrays and combinatorial designs and the construction of most combinatorial designs are unresolved. Thus, the optimal SSDs obtained by their methods are rather limited.

(c) Yamada et al. (2006) presented a method for constructing mixed-level SSDs by juxtaposing two SSDs, each of which is generated by the operation "$\oplus$" of an initial matrix and a generating matrix. It can be seen that their operation "$\oplus$" is in fact equivalent to the "$\oplus_{\mathcal{A}}$" in this paper with $\mathcal{A} = \{0, \dots, q-1\}$, and they only provided the theoretical justification of the χ^2-optimality for the SSD with $n = 6$. Recently, Liu and Lin (2009) proposed a method to construct χ^2-optimal mixed-level SSDs from smaller multi-level SSDs and transposed orthogonal arrays based on Kronecker sums. It can be easily confirmed that the result of Liu and Lin (2009) is merely a special case of our Theorem 3, by taking F_1 as $L_{q_1} = (0, \dots, q_1 - 1)'$ and D as $L_{rq_1}(q_1^{n_2})$. Thus, all their designs can be constructed by our Theorem 3.

(d) Using k-cyclic generators, Chen and Liu (2008a) and Liu and Zhang (2009) constructed some $E(f_{\mathrm{NOD}})$- and χ^2-optimal mixed-level SSDs, respectively. The k-cyclic generators were obtained via computer searches, when the values of k, the run size and/or the level sizes become larger, the computer searches tend to be ineffective and impractical.

(e) Recently, Liu and Cai (2009) proposed a new construction method, called the substitution method, for $E(f_{\mathrm{NOD}})$-optimal SSDs. It can be seen that all the $E(f_{\mathrm{NOD}})$-optimal SSDs tabulated in our Tables 6 and 8 are different from those tabulated in their Appendices.

Note that the newly proposed methods use small equidistant designs and difference matrices to generate large designs. Many difference matrices can be found in Hedayat, Slone and Stufken (1999), Wu and Hamada (2000) and from the site http://support.sas.com/techsup/technote/ts723.html maintained by Dr. W. F. Kuhfeld of SAS. Equidistant designs can be found in Ngugen (1996), Tang and Wu (1997), Liu and Zhang (2000), Lu et al. (2002), Fang, Lin and Liu (2003), Fang, Ge and Liu (2002a, 2002b, 2004), Lu, Hu and Zheng (2003), Fang et al. (2003, 2004a, 2004b), Aggarwal and Gupta (2004), Eskridge et al. (2004), Georgiou and Koukouvinos (2006), Georgiou, Koukouvinos and Mantas (2006), Chen and Liu (2008a), Liu and Cai (2009) and others. Difference matrices can also be obtained from orthogonal arrays or by taking the Kronecker sums of difference matrices, and equidistant designs also include saturated orthogonal arrays of strength two.

The appealing feature of our methods is that they can be easily applied and the resulting designs are $E(f_{\mathrm{NOD}})$- and/or χ^2-optimal SSDs without fully aliased columns. In particular, the nonorthogonality between columns of the resulting designs is well-controlled by the source designs, that is, if the source designs have little nonorthogonality, the generated design will also have little nonorthogonality.

From these proposed methods, many optimal SSDs can be constructed in addition to those tabulated in the Appendix.

In regard to the statistical data analysis for mixed-level SSDs, it should be noted that analyzing the data collected by such SSDs is a very important but complicated task which has attracted much recent attention. See, for example, Zhang, Zhang and Liu (2007), Phoa, Pan and Xu (2009) and Li, Zhao and Zhang (2010). When there are many more factors than the number of permitted runs due to expense (e.g., money or time), the nonorthogonality among factors may be very severe and may prevent the few active factors to be identified correctly by any existing method. Therefore, the data analysis for SSDs in general remains an important and challenging topic for further research. Some recent study on the analysis of "high-dimension and low-sample size" in genetic studies (e.g., studying 6,000 genes with only 37 observations) may be relevant.

APPENDIX A: PROOFS

PROOF OF THEOREM 1. (a)(i) If f^1 and f^3 are fully aliased, that is, f^1 can be obtained by permutating the levels of f^3, then there must exist a permutation matrix Q of order q that satisfies $X_1 = X_3 Q$, thus $X_1 X_1' = X_3 Q Q' X_3' = X_3 X_3'$.

On the other hand, let V_1 and V_3 be the vector spaces spanned by the columns of X_1 and X_3, respectively. If $X_1 X_1' = X_3 X_3'$, then $V_1 = V_3$, and for any column x^0 of X_1, we have

$$x^0 = k_1 x_3^1 + \cdots + k_q x_3^q \qquad \text{where } x_3^i \text{ is the } i\text{th column of } X_3, i = 1, \ldots, q.$$

Since any two columns in an induced matrix share no element 1 at any position and each column has n_1/q ones and $n_1 - n_1/q$ zeros, there must exist only one $k_i \neq 0$, that is, x^0 is identical to a column of X_3. Then there exists a permutation matrix Q of order q satisfying $X_1 = X_3 Q$, thus f^1 can be obtained by permutating the levels of f^3, that is, f^1 and f^3 are fully aliased.

(ii) Note that the induced matrix of $f^1 \oplus_{\mathcal{A}} f^2$ is $[(X_2 P_{f_{11}})', \ldots, (X_2 P_{f_{n_1 1}})']'$ and $P_{f_{i1}} = \sum_{t=1}^{q} x_{it}^1 P_{t-1}$, then

$$\begin{pmatrix} X_2 P_{f_{11}} \\ \vdots \\ X_2 P_{f_{n_1 1}} \end{pmatrix} = \operatorname{diag}\{\underbrace{X_2, \ldots, X_2}_{n_1}\} \begin{pmatrix} \sum_{t=1}^{q} x_{1t}^1 P_{t-1} \\ \vdots \\ \sum_{t=1}^{q} x_{n_1 t}^1 P_{t-1} \end{pmatrix}$$
$$= (I_{n_1} \otimes X_2)(X_1 \otimes I_q) P = (X_1 \otimes X_2) P,$$

where I_n is the identity matrix of order n.

(iii) The induced matrices of $f^1 \oplus_{\mathcal{A}} f^2$ and $f^3 \oplus_{\mathcal{A}} f^4$ are $(X_1 \otimes X_2)P$ and $(X_3 \otimes X_4)P$, respectively. If $f^1 \oplus_{\mathcal{A}} f^2$ and $f^3 \oplus_{\mathcal{A}} f^4$ are fully aliased, then there exists a permutation matrix Q of order q such that

$$(X_1 \otimes X_2)P = (X_3 \otimes X_4)PQ, \qquad \text{that is, } \sum_{t=1}^{q} x_{st}^1 X_2 P_{t-1} = \sum_{t=1}^{q} x_{st}^3 X_4 P_{t-1} Q$$

for $s = 1, \dots, n_1$. For any s and i, there is only one nonzero element of x_{st}^i for $t = 1, \dots, q$ that equals 1. Thus, $X_2 P_{t_1-1} = X_4 P_{t_3-1} Q$, and therefore f^2 is fully aliased with f^4. Similarly since $f^2 \oplus_{\mathcal{A}} f^1$ and $f^4 \oplus_{\mathcal{A}} f^3$ are also fully aliased, it follows that f^1 is fully aliased with f^3.

(b)(i) It can be obtained easily from the definition of an induced matrix.

(ii) From (a)(i), we only need to prove that columns $q_2(f^1 - \frac{q_1-1}{2}) \oplus (f^2 - \frac{q_2-1}{2}) + \frac{q_1q_2-1}{2}$ and $q_4(f^3 - \frac{q_3-1}{2}) \oplus (f^4 - \frac{q_4-1}{2}) + \frac{q_3q_4-1}{2}$ are fully aliased if and only if $X_1X_1' = X_3X_3'$ and $X_2X_2' = X_4X_4'$. From (b)(i), the induced matrices of these two columns are $X_1 \otimes X_2$ and $X_3 \otimes X_4$, respectively, thus from (a)(i), they are fully aliased if and only if $(X_1 \otimes X_2)(X_1 \otimes X_2)' = (X_3 \otimes X_4)(X_3 \otimes X_4)'$, that is, $X_1X_1' \otimes X_2X_2' = X_3X_3' \otimes X_4X_4'$, which means that $X_1X_1' = aX_3X_3'$ and $X_2X_2' = 1/aX_4X_4'$ for some $a \neq 0$. Since the elements in X_iX_i' are all ones and zeros for $i = 1, \dots, 4$, then $a = 1$, that is, $X_1X_1' = X_3X_3'$ and $X_2X_2' = X_4X_4'$.

(iii) The induced matrices of columns $q_2(f^1 - \frac{q_1-1}{2}) \oplus (f^2 - \frac{q_2-1}{2}) + \frac{q_1q_2-1}{2}$ and $f^3 \oplus_{\mathcal{A}} f^4$ are $X_1 \otimes X_2$ and $[(X_4P_{f_{13}})', \dots, (X_4P_{f_{n_13}})']'$, respectively. If these two columns are fully aliased, then $v_{ij}X_2X_2' = X_4P_{f_{i3}}P_{f_{j3}}'X_4'$, where v_{ij} is the (i, j)th entry of X_1X_1', $i, j = 1, \dots, n_1$. Note that v_{ij} can be zero, and hence $v_{ij}X_2X_2'$ can be a zero matrix which contradicts the fact that $X_4P_{f_{i3}}P_{f_{j3}}'X_4'$ cannot be a zero matrix in any case. □

PROOF OF THEOREM 2. (a) Consider the ith and jth rows of $F \oplus_{\mathcal{A}} D'$, $(f_{i_1} \oplus_{\mathcal{A}} d_{i_2})'$ and $(f_{j_1} \oplus_{\mathcal{A}} d_{j_2})'$, where $i = (i_1 - 1)c + i_2$, $j = (j_1 - 1)c + j_2$, $i_1, j_1 = 1, \dots, n$, $i_2, j_2 = 1, \dots, c$, and $i \neq j$, f_k and d_k are the kth rows of F and D', respectively. Then the coincidence number between $(f_{i_1} \oplus_{\mathcal{A}} d_{i_2})'$ and $(f_{j_1} \oplus_{\mathcal{A}} d_{j_2})'$ equals the number of zeros in $(f_{i_1} - f_{j_1}) \oplus_{\mathcal{A}} (d_{i_2} - d_{j_2})$.

(i) Suppose $i_1 = j_1, i_2 \neq j_2$, then $f_{i_1} = f_{j_1}$ and $d_{i_2} \neq d_{j_2}$. From the definition of difference matrix, each element in $\mathcal{A}$ occurs r times in $d_{i_2} - d_{j_2}$. Therefore, $(f_{i_1} - f_{j_1}) \oplus_{\mathcal{A}} (d_{i_2} - d_{j_2}) = 0_m \oplus_{\mathcal{A}} (d_{i_2} - d_{j_2})$ and there are mr zeros in $0_m \oplus_{\mathcal{A}} (d_{i_2} - d_{j_2})$, where 0_m denotes the $m \times 1$ column vector with all elements zero, that is, the coincidence number between $(f_{i_1} \oplus_{\mathcal{A}} d_{i_2})'$ and $(f_{j_1} \oplus_{\mathcal{A}} d_{j_2})'$ is mr.

(ii) If $i_1 \neq j_1, i_2 \neq j_2$, similar to (i), it can also be easily seen that there are mr zeros in $(f_{i_1} - f_{j_1}) \oplus_{\mathcal{A}} (d_{i_2} - d_{j_2})$, that is, the coincidence number between $(f_{i_1} \oplus_{\mathcal{A}} d_{i_2})'$ and $(f_{j_1} \oplus_{\mathcal{A}} d_{j_2})'$ is mr.

(iii) If $i_1 \neq j_1, i_2 = j_2$, that is, $f_{i_1} \neq f_{j_1}$, $d_{i_2} = d_{j_2}$, then $(f_{i_1} - f_{j_1}) \oplus_{\mathcal{A}} (d_{i_2} - d_{j_2}) = (f_{i_1} - f_{j_1}) \oplus_{\mathcal{A}} 0_{rq}$, and there are λrq zeros in $(f_{i_1} - f_{j_1}) \oplus_{\mathcal{A}} 0_{rq}$, that is, the coincidence number between $(f_{i_1} \oplus_{\mathcal{A}} d_{i_2})'$ and $(f_{j_1} \oplus_{\mathcal{A}} d_{j_2})'$ is λrq.

(b) $F \oplus_{\mathcal{A}} D'$ can be obtained from $D' \oplus_{\mathcal{A}} F$ through row and column permutations. Thus, if $D' \oplus_{\mathcal{A}} F$ has no fully aliased columns, neither does $F \oplus_{\mathcal{A}} D'$. Let $d^1 \oplus_{\mathcal{A}} f^1$ and $d^2 \oplus_{\mathcal{A}} f^2$ be two different columns of $D' \oplus_{\mathcal{A}} F$, where $d^i = (d_{1i}, \dots, d_{ci})'$ and $f^i = (f_{1i}, \dots, f_{ni})'$ for $i = 1$ and 2 are columns of D' and F, respectively. Since D is a normalized difference matrix, $d_{1i} = 0$ for $i = 1$ and 2. Let X_1 and X_2 be the induced matrices of f^1 and f^2, respectively. Then the induced matrices of $d^1 \oplus_{\mathcal{A}} f^1$ and $d^2 \oplus_{\mathcal{A}} f^2$ are

$$\begin{pmatrix} X_1 \\ X_1 P_{d_{21}} \\ \vdots \\ X_1 P_{d_{c1}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} X_2 \\ X_2 P_{d_{22}} \\ \vdots \\ X_2 P_{d_{c2}} \end{pmatrix} \qquad \text{respectively.}$$

Suppose $d^1 \oplus_{\mathcal{A}} f^1$ and $d^2 \oplus_{\mathcal{A}} f^2$ are fully aliased. Then from Theorem 1,

$$X_1 P_{d_{1i}} P'_{d_{j1}} X'_1 = X_2 P_{d_{i2}} P'_{d_{j2}} X'_2, \qquad i, j = 1, \dots, c. \tag{3}$$

Noting that $P_{d_{11}} = P_{d_{12}} = I_q$, we can obtain the following equations by taking $i = 1$ in (3):

$$X_1 X'_1 = X_2 X'_2, \tag{4}$$

$$X_1 P'_{d_{j1}} X'_1 = X_2 P'_{d_{j2}} X'_2, \qquad j = 2, \dots, c. \tag{5}$$

Since F has no fully aliased columns, from equation (4), we know that f^1 and f^2 must be the same column of F, thus $X_1 = X_2$, and $X_1 P'_{d_{j1}} X'_1 = X_1 P'_{d_{j2}} X'_1$ for $j = 2, \dots, c$. Also, since X_1 is a column full rank matrix, we have $P'_{d_{j1}} = P'_{d_{j2}}$, and thus $d_{j1} = d_{j2}$, for $j = 1, \dots, c$, that is, $d^1 = d^2$. So d^1 and d^2 must be the same row of D since D has no identical rows. Therefore, $d^1 \oplus_{\mathcal{A}} f^1$ and $d^2 \oplus_{\mathcal{A}} f^2$ are the same column of $D' \oplus_{\mathcal{A}} F$, which contradicts the fact that they are two different columns of $D' \oplus_{\mathcal{A}} F$. Hence, $D' \oplus_{\mathcal{A}} F$ as well as $F \oplus_{\mathcal{A}} D'$ have no fully aliased columns. $\square$

PROOF OF THEOREM 3. We only prove that there are no fully aliased columns between $F_1 \oplus_{\mathcal{A}_1} D'$ and $0_{n_1} \oplus F_2$. (The others can be proved easily.) Suppose $f^1 \oplus_{\mathcal{A}_1} d^1$ and $0_{n_1} \oplus f^2$ are columns of $F_1 \oplus_{\mathcal{A}_1} D'$ and $0_{n_1} \oplus F_2$, respectively, where f^1, f^2 and $d^1 = (0, d_{21}, \dots, d_{n_2 1})'$ are columns of F_1, F_2 and D', respectively. Let X and Y be the induced matrices of f^1 and f^2, respectively. Then the induced matrices of $d^1 \oplus_{\mathcal{A}_1} f^1$ and $f^2 \oplus 0_{n_1}$ are $[X', (X P_{d_{21}})', \dots, (X P_{d_{n_2 1}})']$ and $Y \otimes 1_{n_1}$, respectively. From the definition of an induced matrix, it is easy to see that $XX' \neq y_0 1_{n_1} 1'_{n_1}$, where y_0 is the (1, 1)th entry of YY' and 1_{n_1} denotes the $n_1 \times 1$ vector with all elements unity. Thus, from Theorem 1, $d^1 \oplus_{\mathcal{A}_1} f^1$ and $f^2 \oplus 0_{n_1}$ are not fully aliased. Therefore, $f^1 \oplus_{\mathcal{A}_1} d^1$ and $0_{n_1} \oplus f^2$ are not fully aliased. $\square$

The following lemma will be used in the proof of Theorem 6.

LEMMA 3 [Fang, Lin and Liu (2003)]. *Suppose f^j is the jth column of an $F(n, q_1 \cdots q_m)$ with induced matrix X_j, $j = 1, \dots, m$. Then*

$$f_{\mathrm{NOD}}(f^i, f^j) = \mathrm{tr}(X_i' X_j X_j' X_i) - \frac{n^2}{q_i q_j}.$$

PROOF OF THEOREM 6. (a) From Theorem 1 and Lemma 3, the induced matrices of h_1 and h_2 are $X_1 \otimes X_2$ and $X_3 \otimes X_4$, respectively. Then we have

$$\begin{aligned} f_{\mathrm{NOD}}(h_1, h_2) &= \mathrm{tr}[(X_1' X_3 X_3' X_1) \otimes (X_2' X_4 X_4' X_2)] - \frac{n_1^2 n_2^2}{\prod_{i=1}^4 q_i} \\ &= \mathrm{tr}(X_1' X_3 X_3' X_1)\,\mathrm{tr}(X_2' X_4 X_4' X_2) - \frac{n_1^2 n_2^2}{\prod_{i=1}^4 q_i} \\ &= f_{\mathrm{NOD}}(f^1, f^3) f_{\mathrm{NOD}}(f^2, f^4) \\ &\quad + \frac{n_2^2}{q_2 q_4} f_{\mathrm{NOD}}(f^1, f^3) + \frac{n_1^2}{q_1 q_3} f_{\mathrm{NOD}}(f^2, f^4). \end{aligned}$$

(b) The induced matrices of $f^1 \oplus_{\mathcal{A}_1} f^2$ and $f^3 \oplus_{\mathcal{A}_3} f^4$ are $(X_1 \otimes X_2)P$ and $(X_3 \otimes X_4)Q$, respectively, where $P = (P_0', \dots, P_{q_1-1}')'$, $Q = (Q_0', \dots, Q_{q_3-1}')'$, P_i and Q_j are permutation matrices defined by $(0, \dots, q_1 - 1)P_i' = i +_{\mathcal{A}_1} (0, \dots, q_1 - 1)$ and $(0, \dots, q_3 - 1)Q_j' = j +_{\mathcal{A}_3} (0, \dots, q_3 - 1)$, respectively, $i = 0, \dots, q_1 - 1$, $j = 0, \dots, q_3 - 1$. Let $T = P'(X_1' X_3 \otimes X_2' X_4)Q - \frac{n_1 n_2}{q_1 q_3} 1_{q_1} 1_{q_3}'$. Then from Lemma 3, $f_{\mathrm{NOD}}(f^1 \oplus_{\mathcal{A}_1} f^2, f^3 \oplus_{\mathcal{A}_3} f^4)$ equals the sum of squares of the elements of T. Let $W = (w_{ij}) = X_1' X_3$, $B = (b_{ij}) = X_2' X_4 - n_2/(q_1 q_3) 1_{q_1} 1_{q_3}'$, and note that $\sum_{j=1}^{q_3} \sum_{i=1}^{q_1} w_{ij} = n_1$, $\sum_{j=1}^{q_3} \sum_{i=1}^{q_1} b_{ij} = 0$. Then

$$T = \sum_{j=1}^{q_3} \sum_{i=1}^{q_1} w_{ij} P_{i-1}' B Q_{j-1}$$

and the (s, t)th entry of T can be expressed as $\sum_{j=1}^{q_3} \sum_{i=1}^{q_1} w_{ij} b_{s_i t_j}$, where $(s_1, \dots, s_{q_1})$ and $(t_1, \dots, t_{q_3})$ are some permutations of $(1, \dots, q_1)$ and $(1, \dots, q_3)$, respectively. Then

$$\left(\sum_{j=1}^{q_3} \sum_{i=1}^{q_1} w_{ij} b_{s_i t_j} \right)^2 \le \left(\sum_{j=1}^{q_3} \sum_{i=1}^{q_1} w_{ij}^2 \right) \left(\sum_{j=1}^{q_3} \sum_{i=1}^{q_1} b_{ij}^2 \right),$$

and thus

$$\begin{aligned} f_{\mathrm{NOD}}(f^1 \oplus_{\mathcal{A}_1} f^2, f^3 \oplus_{\mathcal{A}_3} f^4) &\le q_1 q_3 \left(\sum_{j=1}^{q_3} \sum_{i=1}^{q_1} w_{ij}^2 \right) \left(\sum_{j=1}^{q_3} \sum_{i=1}^{q_1} b_{ij}^2 \right) \\ &= q_1 q_3 \left[f_{\mathrm{NOD}}(f^1, f^3) + \frac{n_1^2}{q_1 q_3} \right] f_{\mathrm{NOD}}(f^2, f^4), \end{aligned}$$

where the equality holds if and only if there exist c_1 and c_2 with $|c_1| + |c_2| > 0$ such that $c_1 w_{ij} = c_2 b_{s_i t_j}$ for $i = 1, \dots, q_1$ and $j = 1, \dots, q_3$. This means that $c_1 \sum_{j=1}^{q_3} \sum_{i=1}^{q_1} w_{ij} = c_2 \sum_{j=1}^{q_3} \sum_{i=1}^{q_1} b_{s_i t_j} = 0$, and so $c_1 = 0, c_2 \neq 0$ and $b_{ij} = 0$ for $i = 1, \dots, q_1$ and $j = 1, \dots, q_3$. Thus, f^2 is orthogonal to f^4.

On the other hand, if f^2 is orthogonal to f^4, $f_{\mathrm{NOD}}(f^2, f^4) = 0$ and

$$\begin{aligned} 0 &\leq f_{\mathrm{NOD}}(f^1 \oplus_{\mathcal{A}_1} f^2, f^3 \oplus_{\mathcal{A}_3} f^4) \\ &\leq q_1 q_3 \left[f_{\mathrm{NOD}}(f^1, f^3) + \frac{n_1^2}{q_1 q_3} \right] f_{\mathrm{NOD}}(f^2, f^4) = 0, \end{aligned}$$

then the equality holds.

Similarly, we can obtain that

$$\begin{aligned} f_{\mathrm{NOD}}(f^1 \oplus_{\mathcal{A}_1} f^2, f^3 \oplus_{\mathcal{A}_3} f^4) &= f_{\mathrm{NOD}}(f^2 \oplus_{\mathcal{A}_1} f^1, f^4 \oplus_{\mathcal{A}_3} f^3) \\ &\leq q_2 q_4 \left[f_{\mathrm{NOD}}(f^2, f^4) + \frac{n_2^2}{q_2 q_4} \right] f_{\mathrm{NOD}}(f^1, f^3), \end{aligned}$$

and the equality holds if and only if f^1 is orthogonal to f^3. Hence, we have the assertion.

(c) The induced matrices of $f^1 \oplus_{\mathcal{A}_1} f^2$ and h_2 are $(X_1 \otimes X_2)P$ and $X_3 \otimes X_4$, respectively. Let $K = P'(X_1' X_3 \otimes X_2' X_4) - \frac{n_1 n_2}{q_1 q_3 q_4} 1_{q_1} 1'_{q_3 q_4}$, $G = (g_{ij}) = X_2' X_4 - \frac{n_2}{q_1 q_4} 1_{q_1} 1'_{q_4}$ and $W = (w_{ij}) = X_1' X_3$, and note that $\sum_{i=1}^{q_1} w_{ij} = n_1/q_3$, $j = 1, \dots, q_3$. Then $K = (A_1, \dots, A_{q_3})$, where $A_j = \sum_{i=1}^{q_1} w_{ij} P'_{j-1} G$. Note that $f_{\mathrm{NOD}}(f^1 \oplus_{\mathcal{A}_1} f^2, h_2)$ is equal to the sum of squares of the elements of K, the (s,t)th entry of A_j is $\sum_{i=1}^{q_1} w_{ij} g_{s_i t}$ and $(\sum_{i=1}^{q_1} w_{ij} g_{s_i t})^2 \leq \sum_{i=1}^{q_1} w_{ij}^2 \sum_{s=1}^{q_1} g_{st}^2$, where $(s_1, \dots, s_{q_1})$ is a permutation of $(1, \dots, q_1)$. Then similar to the proof in (b), we get

$$\begin{aligned} f_{\mathrm{NOD}}(f^1 \oplus_{\mathcal{A}_1} f^2, h_2) &\leq \sum_{j=1}^{q_3} \sum_{t=1}^{q_4} \sum_{s=1}^{q_1} \left(\sum_{i=1}^{q_1} w_{ij}^2 \sum_{k=1}^{q_1} g_{kt}^2 \right) \\ &= q_1 \sum_{j=1}^{q_3} \sum_{i=1}^{q_1} w_{ij}^2 \sum_{t=1}^{q_4} \sum_{k=1}^{q_1} g_{kt}^2 \\ &= q_1 \left[f_{\mathrm{NOD}}(f^1, f^3) + \frac{n_1^2}{q_1 q_3} \right] f_{\mathrm{NOD}}(f^2, f^4), \end{aligned}$$

where the equality holds if and only if f^2 is orthogonal to f^4, and

$$f_{\mathrm{NOD}}(f^1 \oplus_{\mathcal{A}_1} f^2, h_2) \leq q_1 \left[f_{\mathrm{NOD}}(f^2, f^4) + \frac{n_2^2}{q_1 q_4} \right] f_{\mathrm{NOD}}(f^1, f^3),$$

where the equality holds if and only if f^1 is orthogonal to f^3. Thus, we complete the proof of (c). $\square$

APPENDIX B: SOME SELECTED OPTIMAL SUPERSATURATED DESIGNS

TABLE 5
Equidistant designs used in Tables 6–9

n	m	q	Source design
4	3	2	Orthogonal array
8	7	2	Orthogonal array
12	11	2	Orthogonal array
16	15	2	Orthogonal array
16	5	4	Orthogonal array
20	19	2	Orthogonal array
24	23	2	Orthogonal array
25	6	5	Orthogonal array
6	10	2	Liu and Zhang (2000)
6	5	3	Fang, Ge and Liu (2004)
6	$5k$ $(k = 2, 3)$	3	Georgiou and Koukouvinos (2006)
8	$7k$ $(k = 2, \dots, 5)$	2	Liu and Zhang (2000)
8	$7k$ $(k = 1, 2)$	4	Fang, Ge and Liu (2002a)
8	$7k$ $(k = 3, \dots, 6)$	4	Georgiou and Koukouvinos (2006)
9	$4k$ $(k = 1, \dots, 7)$	3	Fang, Ge and Liu (2004)
9	$4k$ $(k = 8, 10, 12)$	3	Georgiou and Koukouvinos (2006)
10	$18k$ $(k = 1, 2, 3)$	2	Liu and Zhang (2000)
10	9	5	Fang, Ge and Liu (2002b)
10	$9k$ $(k = 2, 3, 4)$	5	Georgiou and Koukouvinos (2006)
12	$11k$ $(k = 2, \dots, 12)$	2	Liu and Zhang (2000)
12	11	3	Lu, Hu and Zheng (2003)
12	$11k$ $(k = 2, \dots, 5)$	3	Georgiou and Koukouvinos (2006)
12	11	6	Lu, Hu and Zheng (2003)
12	$11k$ $(k = 2, 3)$	6	Georgiou and Koukouvinos (2006)
14	$13k$ $(k = 1, 2)$	7	Fang et al. (2003)
15	28	3	Georgiou and Koukouvinos (2006)
15	$7k$ $(k = 1, \dots, 13)$	5	Fang, Ge and Liu (2004)
16	$15k$ $(k = 2, \dots, 6)$	2	Liu and Zhang (2000)
16	$15k$ $(k = 7, 8, 9)$	2	Eskridge et al. (2004)
16	$5k$ $(k = 2, \dots, 6)$	4	Fang et al. (2003)
16	$5k$ $(k = 7, \dots, 16)$	4	Georgiou, Koukouvinos and Mantas (2006)
18	$34k$ $(k = 1, 2, 3)$	2	Liu and Zhang (2000)
18	$17k$ $(k = 1, 2)$	3	Fang et al. (2003)
18	17	6	Lu, Hu and Zheng (2003)
18	34	6	Georgiou and Koukouvinos (2006)
20	$19k$ $(k = 2, 3)$	2	Liu and Zhang (2000)
20	19	4	Lu et al. (2002)
20	19	5	Lu et al. (2002)
22	42	2	Liu and Zhang (2000)
24	46	2	Liu and Zhang (2000)
24	23	4	Lu et al. (2002)
24	23	6	Lu, Hu and Zheng (2003)
25	$6k$ $(k = 2, \dots, 25)$	5	Georgiou, Koukouvinos and Mantas (2006)

TABLE 6
Some selected $E(f_{\mathrm{NOD}})$-optimal SSDs constructed by Theorem 3

n_1	m_1	q_1	n_2	m_2	q_2	r	**Final resulting SSD**†	λ
4	3	2	6	$5k$	3	$4k$	$F(24, 2^{24k}3^{5k})$	$13k,\ k=1,2,3$
4	3	2	8	$7k$	4	$6k$	$F(32, 2^{36k}4^{7k})$	$19k,\ k=1,\dots,6$
4	3	2	9	$4k$	3	$3k$	$F(36, 2^{18k}3^{4k})$	$10k,\ k=2t, t=1\dots,6$
6	5	3	6	10	2	3	$F(36, 2^{10}3^{45})$	19
4	3	2	10	$9k$	5	$8k$	$F(40, 2^{48k}5^{9k})$	$25k,\ k=1,\dots,4$
4	3	2	12	$11k$	3	$8k$	$F(48, 2^{48k}3^{11k})$	$27k,\ k=1,\dots,5$
6	5	3	8	$7k$	2	$2k$	$F(48, 2^{7k}3^{30k})$	$13k,\ k=2,\dots,5$
6	10	2	8	$7k$	4	$3k$	$F(48, 2^{60k}4^{7k})$	$31k,\ k=2,4,6$
4	3	2	12	$11k$	6	$10k$	$F(48, 2^{60k}6^{11k})$	$31k,\ k=1,2,3$
6	5	3	8	$7k$	4	$3k$	$F(48, 3^{45k}4^{7k})$	$16k,\ k=1,\dots,6$
6	10	3	8	$7k$	2	k	$F(48, 2^{7k}3^{30k})$	$13k,\ k=3,4,5$
6	10	3	8	$14k$	4	$3k$	$F(48, 3^{90k}4^{14k})$	$32k,\ k=1,2,3$
6	10	2	9	16	3	6	$F(54, 2^{120}3^{16})$	64
4	3	2	14	$13k$	7	$12k$	$F(56, 2^{72k}7^{13k})$	$37k,\ k=1,2$
4	3	2	15	28	3	20	$F(60, 2^{120}3^{28})$	68
6	5	3	10	$18k$	2	$5k$	$F(60, 2^{18k}3^{75k})$	$33k,\ k=2,3$
4	3	2	15	$7k$	5	$6k$	$F(60, 2^{36k}5^{7k})$	$19k,\ k=2,\dots,13$
6	10	2	10	$9k$	5	$4k$	$F(60, 2^{80k}5^{9k})$	$41k,\ k=2,3,4$
6	5	3	10	$9k$	5	$4k$	$F(60, 3^{60k}5^{9k})$	$21k,\ k=1,\dots,4$
10	9	5	6	$5k$	3	k	$F(60, 3^{5k}5^{45k})$	$10k,\ k=2,3$
6	10	3	10	18	5	4	$F(60, 3^{120}5^{18})$	42
4	3	2	16	$5k$	4	$4k$	$F(64, 2^{24k}4^{5k})$	$13k,\ k=2,\dots,16$
4	3	2	18	$17k$	3	$12k$	$F(72, 2^{72k}3^{17k})$	$41k,\ k=1,2$
6	10	2	12	$11k$	3	$4k$	$F(72, 2^{80k}3^{11k})$	$43k,\ k=2,\dots,5$
4	3	2	18	34	6	30	$F(72, 2^{180}6^{34})$	94
6	10	2	12	22	6	10	$F(72, 2^{200}6^{22})$	102
9	4	3	8	$7k$	4	$6k$	$F(72, 3^{72k}4^{7k})$	$25k,\ k=1,\dots,6$
6	5	3	12	$11k$	6	$5k$	$F(72, 3^{75k}6^{11k})$	$26k,\ k=2,3$
10	18	2	6	$5k$	3	$2k$	$F(80, 2^{72k}3^{5k})$	$37k,\ k=2,3$
10	18	2	8	$7k$	4	$3k$	$F(80, 2^{108k}4^{7k})$	$55k,\ k=2,4,6$
4	3	2	20	19	5	16	$F(80, 2^{96}5^{19})$	51
10	9	5	8	$14k$	4	$3k$	$F(80, 4^{14k}5^{135k})$	$29k,\ k=1,2,3$
6	10	2	14	26	7	12	$F(84, 2^{240}7^{26})$	122
6	5	3	14	26	7	12	$F(84, 3^{180}7^{26})$	62
6	10	2	15	$7k$	5	$3k$	$F(90, 2^{60k}5^{7k})$	$31k,\ k=2t, t=2,\dots,6$
6	5	3	15	$7k$	5	$3k$	$F(90, 3^{45k}5^{7k})$	$16k,\ k=2\dots,13$
9	4	3	10	$9k$	5	$8k$	$F(90, 3^{96k}5^{9k})$	$33k,\ k=1,\dots,4$
6	5	3	16	$15k$	2	$4k$	$F(96, 2^{15k}3^{60k})$	$27k,\ k=2,\dots,9$
4	3	2	24	23	4	18	$F(96, 2^{108}4^{23})$	59
6	10	2	16	$5k$	4	$2k$	$F(96, 2^{40k}4^{5k})$	$21k,\ k=4,\dots,16$
4	3	2	24	23	6	20	$F(96, 2^{120}6^{23})$	63
6	5	3	16	$5k$	4	$2k$	$F(96, 3^{30k}4^{5k})$	$11k,\ k=3,\dots,16$
4	3	2	25	$6k$	5	$5k$	$F(100, 2^{30k}5^{6k})$	$16k,\ k=2t, t=2,\dots,12$

† $F(n_1n_2, q_1^{rm_1q_1}q^{m_2})$.

λ is the constant coincidence number of the final resulting SSD.

TABLE 7
Some selected χ^2-optimal SSDs constructed by Theorem 3

n_1	m_1	q_1	n_2	m_2	q_2	r	**Final resulting SSD**†	ω
4	3	2	6	$5k$	3	$6k$	$F(24, 2^{36k}3^{5k})$	$39k,\ k=1,2,3$
4	3	2	8	$7k$	4	$12k$	$F(32, 2^{72k}4^{7k})$	$76k,\ k=1,\dots,6$
4	3	2	9	$16k$	3	$18k$	$F(36, 2^{108k}3^{16k})$	$120k,\ k=1,2,3$
6	10	2	6	10	3	6	$F(36, 2^{120}3^{10})$	126
6	5	3	6	10	2	2	$F(36, 2^{10}3^{30})$	38
4	3	2	10	$9k$	5	$20k$	$F(40, 2^{120k}5^{9k})$	$125k,\ k=1,\dots,4$
4	3	2	12	$11k$	3	$12k$	$F(48, 2^{72k}3^{11k})$	$81k,\ k=1,\dots,5$
8	14	2	6	10	3	6	$F(48, 2^{168}3^{10})$	174
4	3	2	12	$11k$	6	$30k$	$F(48, 2^{180k}6^{11k})$	$186k,\ k=1,2,3$
6	10	2	8	$7k$	4	$6k$	$F(48, 2^{120k}4^{7k})$	$124k,\ k=1,\dots,6$
6	5	3	8	$7k$	4	$4k$	$F(48, 3^{60k}4^{7k})$	$64k,\ k=1,\dots,6$
6	10	3	8	$7k$	4	$2k$	$F(48, 3^{60k}4^{7k})$	$64k,\ k=2,\dots,6$
6	15	3	8	42	4	8	$F(48, 3^{360}4^{42})$	384
8	7	2	6	$5k$	3	$6k$	$F(48, 2^{84k}3^{5k})$	$87k,\ k=1,2,3$
8	7	4	6	$5k$	3	k	$F(48, 4^{28k}3^{5k})$	$31k,\ k=2,3$
4	3	2	14	13	7	42	$F(56, 2^{252}7^{13})$	259
4	3	2	15	28	3	30	$F(60, 2^{180}3^{28})$	204
4	3	2	15	14	5	30	$F(60, 2^{180}5^{14})$	190
6	10	2	10	$9k$	5	$10k$	$F(60, 2^{200k}5^{9k})$	$205k,\ k=1,\dots,4$
10	18	2	6	10	3	6	$F(60, 2^{216}3^{10})$	222
4	3	2	16	$5k$	4	$8k$	$F(64, 2^{48k}4^{5k})$	$52k,\ k=2,\dots,16$
8	14	2	8	$7k$	4	$6k$	$F(64, 2^{168k}4^{7k})$	$172k,\ k=1,\dots,6$
4	3	2	18	$17k$	3	$18k$	$F(72, 2^{108k}3^{17k})$	$123k,\ k=1,2$
8	7	2	9	$8k$	3	$9k$	$F(72, 2^{126k}3^{8k})$	$132k,\ k=2,4,6$
9	8	3	8	$7k$	4	$4k$	$F(72, 3^{96k}4^{7k})$	$100k,\ k=1,\dots,6$
9	16	3	8	$7k$	4	$2k$	$F(72, 3^{96k}4^{7k})$	$100k,\ k=2,\dots,6$
6	5	3	12	$11k$	6	$10k$	$F(72, 3^{150k}6^{11k})$	$156k,\ k=1,2,3$
8	7	2	10	$9k$	5	$20k$	$F(80, 2^{280k}5^{9k})$	$285k,\ k=1,\dots,4$
10	18	2	8	$7k$	4	$6k$	$F(80, 2^{216k}4^{7k})$	$220k,\ k=1,\dots,6$
6	5	3	14	$13k$	7	$14k$	$F(84, 3^{210k}7^{13k})$	$217k,\ k=1,2$
6	5	3	15	$7k$	5	$5k$	$F(90, 3^{75k}2^{7k})$	$80k,\ k=2,\dots,11$
6	10	3	16	30	4	8	$F(96, 3^{240}4^{30})$	264
10	18	2	10	$9k$	5	$10k$	$F(100, 2^{360k}2^{9k})$	$365k,\ k=1,\dots,4$
9	4	3	12	$11k$	2	$4k$	$F(108, 3^{48k}2^{11k})$	$58k,\ k=1,\dots,12$
10	18	2	12	$11k$	3	$6k$	$F(120, 2^{216k}3^{11k})$	$225k,\ k=1,\dots,5$
8	7	4	14	26	7	14	$F(112, 4^{392}4^{26})$	406
8	14	2	16	$5k$	4	$4k$	$F(128, 2^{112k}4^{5k})$	$116k,\ k=2,\dots,16$
9	4	3	15	$7k$	5	$10k$	$F(135, 3^{120k}5^{7k})$	$125k,\ k=1,\dots,5$

† $F(n_1n_2, q_1^{rm_1q_1}q^{m_2})$.

ω is the constant natural weighted coincidence number of the final resulting SSD.

TABLE 8
Some selected $E(f_{\mathrm{NOD}})$-optimal SSDs constructed by Theorem 5

n_1	m_1	q_1	n_2	m_2	q_2	m_3	q_3	m_4	q_4	r_3	r_4	**Final resulting SSD**†	λ
4	3	2	6	10	2	3	2	5	3	6	4	$F(24, 2^{36}3^{60}4^{30})$	42
4	3	2	6	10	3	3	2	5	3	8	2	$F(24, 2^{48}3^{30}6^{30})$	36
4	3	2	6	15	3	3	2	5	3	12	3	$F(24, 2^{72}3^{45}6^{45})$	54
4	3	2	8	$7k$	2	3	2	7	4	$4k$	$2k$	$F(32, 2^{24k}4^{77k})$	$29k,\ k=1,\dots,5$
4	3	2	8	21	2	3	2	14	4	12	3	$F(32, 2^{72}4^{231})$	87
4	3	2	8	28	2	3	2	7	4	16	8	$F(32, 2^{96}4^{308})$	116
4	3	2	8	21	4	3	2	7	4	18	2	$F(32, 2^{108}4^{56}8^{63})$	71
6	10	2	6	10	2	10	2	5	3	12	12	$F(36, 2^{240}3^{180}4^{100})$	196
4	3	2	9	$8k$	3	3	2	4	3	$6k$	$4k$	$F(36, 2^{36k}3^{48k}6^{24k})$	$36k,\ k=1,\dots,6$
6	10	2	6	$5k$	3	10	2	5	3	$8k$	$3k$	$F(36, 2^{160k}3^{45k}6^{50k})$	$99k,\ k=1,2,3$
6	10	2	6	10	3	10	3	10	2	8	6	$F(36, 2^{120}3^{240}6^{100})$	148
6	5	3	6	10	2	5	3	10	2	3	8	$F(36, 2^{160}3^{45}6^{50})$	99
6	5	3	6	$5k$	3	10	2	5	3	$2k$	$2k$	$F(36, 2^{40k}3^{30k}9^{25k})$	$31k,\ k=2,3$
6	5	3	6	15	3	10	3	10	2	3	6	$F(36, 2^{120}3^{90}9^{75})$	93
6	10	2	8	7	2	10	3	7	2	4	18	$F(48, 2^{252}3^{120}4^{70})$	178
6	5	3	8	14	2	5	3	14	2	4	12	$F(48, 2^{336}3^{60}6^{70})$	194
6	5	3	8	21	2	10	3	14	2	3	18	$F(48, 2^{504}3^{90}6^{105})$	291
6	5	3	8	28	2	5	3	21	2	8	16	$F(48, 2^{672}3^{120}6^{140})$	388
6	5	3	8	$7k$	4	5	3	7	2	$3k$	$4k$	$F(48, 2^{56k}3^{45k}12^{35k})$	$44k,\ k=1,\dots,6$
6	10	2	8	$7k$	2	10	2	14	4	$8k$	$3k$	$F(48, 2^{160k}4^{238k})$	$134k,\ k=1,\dots,5$
6	5	3	8	$7k$	2	10	2	7	4	$2k$	$4k$	$F(48, 2^{40k}4^{112k}6^{35k})$	$51k,\ k=2,\dots,5$
6	$5k$	3	8	21	2	10	2	$7k$	4	$6k$	12	$F(48, 2^{120k}4^{336k}6^{105k})$	$153k,\ k=1,2,3$
6	10	2	8	$7k$	2	5	3	7	4	$8k$	$6k$	$F(48, 3^{120k}4^{238k})$	$94k,\ k=1,\dots,5$
6	5	3	8	14	2	5	3	7	4	4	8	$F(48, 3^{60}4^{224}6^{70})$	82
6	$5k$	3	8	21	2	$5k$	3	$7k$	4	6	12	$F(48, 3^{90k}4^{336k}6^{105k})$	$123k,\ k=1,2,3$
6	10	3	9	8	3	10	2	8	3	6	8	$F(54, 2^{120}3^{192}9^{80})$	128
10	18	5	6	10	2	18	2	5	3	6	32	$F(60, 2^{216}3^{480}10^{180})$	276
6	10	3	10	9	5	5	3	18	2	8	4	$F(60, 2^{144}3^{120}15^{90})$	114
6	10	3	10	9	5	5	3	9	5	8	2	$F(60, 3^{120}5^{90}15^{90})$	60
8	7	2	8	$7k$	2	7	2	7	4	$12k$	$4k$	$F(64, 2^{168k}4^{161k})$	$121k,\ k=1,\dots,5$
8	$7k$	2	8	14	2	14	2	7	4	$12k$	$8k$	$F(64, 2^{336k}4^{322k})$	$242k,\ k=1,\dots,5$
8	7	2	8	$7k$	4	21	4	7	2	$2k$	$4k$	$F(64, 2^{56k}4^{168k}8^{49k})$	$73k,\ k=1,\dots,6$
8	$7k$	2	8	14	4	21	4	14	2	$4k$	$4k$	$F(64, 2^{112k}4^{336k}8^{98k})$	$146k,\ k=1,\dots,5$
8	$7k$	2	8	21	4	21	2	7	4	$18k$	$4k$	$F(64, 2^{756k}4^{112k}8^{147k})$	$415k,\ k=1,\dots,5$
8	$7k$	2	9	16	3	$7k$	4	16	3	12	$4k$	$F(72, 3^{192k}4^{336k}6^{112k})$	$160k,\ k=1,\dots,5$
10	$9k$	5	8	14	2	$9k$	5	$7k$	4	2	16	$F(80, 4^{448k}5^{90k}10^{126k})$	$136k,\ k=1,\dots,4$
10	$9k$	5	8	21	2	$9k$	5	$14k$	4	3	12	$F(80, 4^{672k}5^{135k}10^{189k})$	$204k,\ k=1,2,3$
10	$9k$	5	8	14	4	$9k$	5	$7k$	2	3	16	$F(80, 2^{224k}5^{135k}20^{126k})$	$141k,\ k=1,\dots,4$

† $F(n_1n_2, (q_1q_2)^{m_1m_2}q_3^{m_3r_3q_3}q_4^{m_4r_4q_4})$; $n_1 = n_3$, $n_2 = n_4$.

λ is the constant coincidence number of the final resulting SSD.

TABLE 9
Some selected χ^2-optimal SSDs constructed by Theorem 5

n_1	m_1	q_1	n_2	m_2	q_2	m_3	q_3	m_4	q_4	r_3	r_4	**Final resulting SSD**†	ω
4	3	2	6	5	3	3	2	5	3	12	2	$F(24, 2^{72}3^{30}6^{15})$	108
4	3	2	8	$7k$	2	3	2	7	4	$8k$	$2k$	$F(32, 2^{48k}4^{77k})$	$116k,\ k = 1, \dots, 5$
9	$8k$	3	4	3	2	$8k$	3	3	2	4	$18k$	$F(36, 2^{108k}3^{96k}6^{24k})$	$216k,\ k = 1, \dots, 6$
6	$5k$	3	6	10	3	10	2	10	3	$18k$	$6k$	$F(36, 2^{360k}3^{180k}9^{50k})$	$558k,\ k = 1, 2, 3$
6	10	2	8	7	2	10	2	14	4	16	3	$F(48, 2^{320}4^{238})$	536
6	5	3	8	7	2	10	2	14	4	6	3	$F(48, 2^{120}4^{168}6^{35})$	306
6	5	3	8	7	2	5	3	14	2	4	18	$F(48, 2^{504}3^{60}6^{35})$	582
6	5	3	8	$7k$	2	5	3	$14k$	4	$4k$	3	$F(48, 3^{60k}4^{168k}6^{35k})$	$246k,\ k = 1, 2, 3$
6	5	3	8	14	2	10	2	7	4	12	12	$F(48, 2^{240}4^{336}6^{70})$	612
6	5	3	8	14	2	5	3	7	4	8	12	$F(48, 3^{120}4^{336}6^{70})$	492
6	5	3	8	21	2	10	2	42	4	18	3	$F(48, 2^{360}4^{504}6^{105})$	918
6	5	3	8	28	2	10	3	42	4	8	4	$F(48, 3^{240}4^{672}6^{140})$	984
6	5	3	8	7	4	10	2	7	4	18	4	$F(48, 2^{360}4^{112}12^{35})$	484
6	5	3	8	7	4	5	3	14	2	12	12	$F(48, 2^{336}3^{180}12^{35})$	528
6	5	3	8	7	4	5	3	7	4	12	4	$F(48, 3^{180}4^{112}12^{35})$	304
6	5	3	8	14	4	10	3	21	2	12	16	$F(48, 2^{672}3^{360}12^{70})$	1056
6	5	3	8	$7k$	4	10	3	7	4	$6k$	$4k$	$F(48, 3^{180k}4^{112k}12^{35k})$	$304k,\ k = 1, \dots, 6$
6	5	3	8	21	4	10	3	7	4	18	12	$F(48, 3^{540}4^{336}12^{105})$	912
6	10	2	9	4	3	10	2	4	3	18	12	$F(54, 2^{360}3^{144}6^{40})$	528
10	$9k$	5	6	5	3	$9k$	5	$5k$	3	3	20	$F(60, 3^{300k}5^{135k}15^{45k})$	$450k,\ k = 1, 2, 3$
8	7	2	8	7	2	14	2	7	4	12	4	$F(64, 2^{336}4^{161})$	484
8	7	2	8	14	2	21	2	7	4	16	8	$F(64, 2^{672}4^{322})$	968
8	7	2	8	21	2	28	2	7	4	18	12	$F(64, 2^{1008}4^{483})$	1452
8	7	2	8	$7k$	4	$7k$	4	$7k$	2	12	16	$F(64, 2^{224k}4^{336k}8^{49k})$	$584k,\ k = 1, \dots, 5$
8	7	2	8	21	4	21	4	28	2	12	12	$F(64, 2^{672}4^{1008}8^{147})$	1752
8	7	2	9	8	3	21	2	8	3	18	8	$F(72, 2^{756}3^{192}6^{56})$	984
8	7	2	9	$8k$	3	$21k$	4	16	3	3	$4k$	$F(72, 3^{192k}4^{252k}6^{56k})$	$480k,\ k = 1, 2$
8	7	4	9	4	3	7	2	8	3	18	12	$F(72, 2^{252}3^{288}12^{28})$	552
8	7	4	9	$4k$	3	$7k$	4	$8k$	3	3	12	$F(72, 3^{288k}4^{84k}12^{28k})$	$384k,\ k = 1, \dots, 6$
9	$8k$	3	8	7	2	$8k$	3	21	2	8	$18k$	$F(72, 2^{756k}3^{192k}6^{56k})$	$984k,\ k = 1, \dots, 6$
9	$8k$	3	8	7	2	$8k$	3	21	4	8	$3k$	$F(72, 3^{192k}4^{252k}6^{56k})$	$480k,\ k = 1, \dots, 6$
9	$8k$	3	8	7	4	16	3	21	2	$12k$	$12k$	$F(72, 2^{504k}3^{576k}12^{56k})$	$1104k,\ k = 1, \dots, 4$
9	$8k$	3	8	7	4	16	3	14	4	$12k$	$3k$	$F(72, 3^{576k}4^{168k}12^{56k})$	$768k,\ k = 1, \dots, 4$
10	9	5	8	$7k$	2	18	2	$7k$	4	$10k$	20	$F(80, 2^{360k}4^{560k}10^{63k})$	$950k,\ k = 1, \dots, 5$
8	14	2	10	$9k$	5	28	2	$9k$	5	$12k$	2	$F(80, 2^{672k}5^{90k}10^{126k})$	$360k,\ k = 1, \dots, 4$
10	$9k$	5	8	7	4	$9k$	5	28	2	6	$20k$	$F(80, 2^{1120k}5^{270k}20^{63k})$	$1410k,\ k = 1, \dots, 5$
8	$7k$	2	10	9	5	$7k$	4	9	5	20	$2k$	$F(80, 4^{560k}5^{90k}10^{63k})$	$680k,\ k = 1, \dots, 5$

† $F(n_1 n_2, (q_1 q_2)^{m_1 m_2} q_3^{m_3 r_3 q_3} q_4^{m_4 r_4 q_4})$; $n_1 = n_3$, $n_2 = n_4$.
ω is the constant natural weighted coincidence number of the final resulting SSD.

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F. SUN
DEPARTMENT OF STATISTICS
KLAS AND SCHOOL OF MATHEMATICS
AND STATISTICS
NORTHEAST NORMAL UNIVERSITY
CHANGCHUN 130024
CHINA
E-MAIL: sfxsfx2001@gmail.com

D. K. J. LIN
DEPARTMENT OF STATISTICS
PENNSYLVANIA STATE UNIVERSITY
UNIVERSITY PARK, PENNSYLVANIA 16802
USA
E-MAIL: DKL5@psu.edu

M.-Q. LIU
DEPARTMENT OF STATISTICS
SCHOOL OF MATHEMATICAL SCIENCES
AND LPMC
NANKAI UNIVERSITY
TIANJIN 300071
CHINA
E-MAIL: mqliu@nankai.edu.cn